Let $\vec{R}$ be the parametric surface in $\mathbb{R}^3$ representing a sphere:

$$\vec{R}(\theta, \phi) = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi)$$

for $-\pi \leq \theta \leq \pi$, $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$ and r be its radius.

$\vec{R}$ can be generalized to a sphere-ish surface where its radius is not constant by all of the parameters (θ and/or ϕ), thus giving:

$$\vec{R}(\theta, \phi) = (r(\theta, \phi)\cos\theta\cos\phi, r(\theta, \phi)\sin\theta\cos\phi, r(\theta, \phi)\sin\phi)$$

for $-\pi \leq \theta \leq \pi$, $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$ and $r(\theta, \phi)$ be an arbitrary application taking (θ, ϕ) to the radius of the point with those coordinates.

Let S be the surface represented by $\vec{R}$, its surface area is calculated by:

$$A = \iint_S \mathrm{d}S$$

Where $dS = ||\vec{N}||\mathrm{d}\phi\mathrm{d}\theta$, begin $\vec{N}$ the normal vector to the surface S. $\vec{N}$ can be computed as:

$$\vec{N} = \frac{\partial \vec{R}}{\partial \theta} \times \frac{\partial \vec{R}}{\partial \phi} \tag{1}$$

It will subsequently be proved that

$$\mathrm{d}S = r(\theta, \phi)\sqrt{\left(\frac{\partial r}{\partial \theta}\right)^2 + \left(r^2(\theta, \phi) + \left(\frac{\partial r}{\partial \phi}\right)^2\right)\cos^2(\phi)}\ \mathrm{d}\phi\mathrm{d}\theta$$

Proof. Note: the following notation will be used:

$$\partial_{x_i} f = \frac{\partial f}{\partial x_i}$$

Let's start by giving a first value to $\vec{N}$ by calculating $\partial_\theta \vec{R}$ and $\partial_\phi \vec{R}$

$$\frac{\partial \vec{R}}{\partial \theta} = \left(\left(\frac{\partial r}{\partial \theta}\cos\theta - r\sin\theta\right)\cos\phi, \left(\frac{\partial r}{\partial \theta}\sin\theta + r\cos\theta\right)\cos\phi, \frac{\partial r}{\partial \theta}\sin\phi\right)$$

$$\frac{\partial \vec{R}}{\partial \phi} = \left(\left(\frac{\partial r}{\partial \phi}\cos\phi - r\sin\phi\right)\cos\theta, \left(\frac{\partial r}{\partial \phi}\cos\phi - r\sin\phi\right)\sin\theta, \frac{\partial r}{\partial \phi}\sin\phi + r\cos\phi\right)$$

Then, by (1)

$$\begin{gathered}\vec{N} = ((\partial_\theta r\sin\theta + r\cos\theta)\cos\phi(\partial_\phi r\sin\phi + r\cos\phi) - \partial_\theta r\sin\phi(\partial_\phi r\cos\phi - r\sin\phi)\sin\theta, \\ \partial_\theta r\sin\phi(\partial_\phi r\cos\phi - r\sin\phi)\cos\theta - (\partial_\phi r\sin\phi + r\cos\phi)(\partial_\theta r\cos\theta - r\sin\theta)\cos\phi, \\ (\partial_\theta r\cos\theta - r\sin\theta)\cos\phi(\partial_\phi r\cos\phi - r\sin\phi)\sin\theta - (\partial_\theta r\sin\theta + r\cos\theta)\cos\phi(\partial_\phi r\cos\phi - r\sin\phi)\cos\theta)\end{gathered} \tag{2}$$

So the square of the euclidean norm of $\vec{N}$ is

$$\begin{gathered}||\vec{N}||^2 = ((\partial_\theta r\sin\theta + r\cos\theta)\cos\phi(\partial_\phi r\sin\phi + r\cos\phi) - \partial_\theta r\sin\phi(\partial_\phi r\cos\phi - r\sin\phi)\sin\theta)^2 \\ +(\partial_\theta r\sin\phi(\partial_\phi r\cos\phi - r\sin\phi)\cos\theta - (\partial_\phi r\sin\phi + r\cos\phi)(\partial_\theta r\cos\theta - r\sin\theta)\cos\phi)^2 \\ +((\partial_\theta r\cos\theta - r\sin\theta)\cos\phi(\partial_\phi r\cos\phi - r\sin\phi)\sin\theta - (\partial_\theta r\sin\theta + r\cos\theta)\cos\phi(\partial_\phi r\cos\phi - r\sin\phi)\cos\theta)^2\end{gathered}$$

The following steps will consist of algebraic manipulation until a *nice* expression for $||\vec{N}||$ is found.

Applying the numbers squared identities to the first and second components of $\vec{N}$, and replacing the third component to $\vec{N}_z^2$ it is obtained

$$\begin{gathered}||\vec{N}||^2 = (\partial_\theta r\sin\theta + r\cos\theta)^2\cos^2\phi(\partial_\phi r\sin\phi + r\cos\phi)^2 + (\partial_\theta r)^2\sin^2\phi(\partial_\phi r\cos\phi - r\sin\phi)^2\sin^2\theta \\ -2(\partial_\theta r\sin\theta + r\cos\theta)\cos\phi(\partial_\phi r\sin\phi + r\cos\phi)\partial_\theta r\sin\phi(\partial_\phi r\cos\phi - r\sin\phi)\sin\theta \\ +(\partial_\theta r)^2\sin^2\phi(\partial_\phi r\cos\phi - r\sin\phi)^2\cos^2\theta + (\partial_\phi r\sin\phi + r\cos\phi)^2(\partial_\theta r\cos\theta - r\sin\theta)^2\cos^2\phi \\ -2\partial_\theta r\sin\phi(\partial_\phi r\cos\phi - r\sin\phi)\cos\theta(\partial_\phi r\sin\phi + r\cos\phi)(\partial_\theta r\cos\theta - r\sin\theta)\cos\phi \\ +\vec{N}_z^2\end{gathered}$$

Now reordering the terms by common factors:

$$\begin{gathered}||\vec{N}||^2 = (\partial_\theta r)^2\sin^2\phi(\partial_\phi r\cos\phi - r\sin\phi)^2\left(\sin^2\theta + \cos^2\theta\right) \\ + \cos^2\phi(\partial_\phi r\sin\phi + r\cos\phi)^2\left((\partial_\theta r\sin\theta + r\cos\theta)^2 + (\partial_\theta r\cos\theta - r\sin\theta)^2\right) \\ -2\partial_\theta r\sin\phi\cos\phi(\partial_\phi r\sin\phi + r\cos\phi)(\partial_\phi r\cos\phi - r\sin\phi)((\partial_\theta r\sin\theta + r\cos\theta)\sin\theta + (\partial_\theta r\cos\theta - r\sin\theta)\cos\theta) \\ +\vec{N}_z^2\end{gathered} \tag{3}$$

Simplifying the additions of some terms of (3)

$$\begin{aligned}(\partial_\theta r\sin\theta + r\cos\theta)^2 + (\partial_\theta r\cos\theta - r\sin\theta)^2 &= (\partial_\theta r)^2\sin^2\theta + r^2\cos^2\theta + 2\partial_\theta r\sin\theta r\cos\theta) \\ &+(\partial_\theta r)^2\cos^2\theta + r^2\sin^2\theta - 2\partial_\theta r\cos\theta r\sin\theta \\ &= \left((\partial_\theta r)^2 + r^2\right)\left(\sin^2\theta + \cos^2\theta\right) = (\partial_\theta r)^2 + r^2\end{aligned} \tag{4}$$

$$\begin{aligned}&(\partial_\theta r\sin\theta + r\cos\theta)\sin\theta + (\partial_\theta r\cos\theta - r\sin\theta)\cos\theta \\ &= \partial_\theta r\sin^2\theta + r\cos\theta\sin\theta + \partial_\theta r\cos^2\theta - r\sin\theta\cos\theta \\ &= \partial_\theta r(\sin^2\theta + \cos^2\theta) = \partial_\theta r\end{aligned} \tag{5}$$

Now let's take a look at $\vec{N}_z$. Since by (2) we know the value of the z (third) component of $\vec{N}$:

$$\vec{N}_z = (\partial_\theta r\cos\theta - r\sin\theta)\cos\phi(\partial_\phi r\cos\phi - r\sin\phi)\sin\theta - (\partial_\theta r\sin\theta + r\cos\theta)\cos\phi(\partial_\phi r\cos\phi - r\sin\phi)\cos\theta$$

Which can also be ordered as

$$\vec{N}_z = \cos\phi(\partial_\phi r\cos\phi - r\sin\phi)((\partial_\theta r\cos\theta - r\sin\theta)\sin\theta - (\partial_\theta r\sin\theta + r\cos\theta)\cos\theta) \tag{6}$$

Simplifying the subtraction on the right

$$(\partial_\theta r\cos\theta - r\sin\theta)\sin\theta - (\partial_\theta r\sin\theta + r\cos\theta)\cos\theta = (\partial_\theta r\cos\theta\sin\theta - r\sin^2\theta) - (\partial_\theta r\sin\theta\cos\theta + r\cos^2\theta)$$

$$= \partial_\theta r\cos\theta\sin\theta - r\sin^2\theta - \partial_\theta r\sin\theta\cos\theta - r\cos^2\theta = (-r)\left(\sin^2\theta + \cos^2\theta\right) = -r$$

Then by (6) $\vec{N}_z^2$ can be written as

$$\vec{N}_z^2 = r^2\cos^2\phi(\partial_\phi r\cos\phi - r\sin\phi)^2 \tag{7}$$

Substituting (4), (5) and (7) to (3) we arrive at

$$\begin{gathered}||\vec{N}||^2 = (\partial_\theta r)^2\sin^2\phi(\partial_\phi r\cos\phi - r\sin\phi)^2 \\ + \cos^2\phi(\partial_\phi r\sin\phi + r\cos\phi)^2\left((\partial_\theta r)^2 + r^2\right) \\ -2(\partial_\theta r)^2\sin\phi\cos\phi(\partial_\phi r\sin\phi + r\cos\phi)(\partial_\phi r\cos\phi - r\sin\phi) \\ +r^2\cos^2\phi(\partial_\phi r\cos\phi - r\sin\phi)^2\end{gathered}$$

$$\begin{gathered}= \left((\partial_\theta r)^2\sin^2\phi + r^2\cos^2\phi\right)(\partial_\phi r\cos\phi - r\sin\phi)^2 \\ + \cos^2\phi(\partial_\phi r\sin\phi + r\cos\phi)^2\left((\partial_\theta r)^2 + r^2\right) \\ -2(\partial_\theta r)^2\sin\phi\cos\phi(\sin\phi\cos\phi((\partial_\phi r)^2 - r^2) + r\partial_\phi r(\cos^2\phi - \sin^2\phi))\end{gathered}$$

To get the end result some more square identities will be applied

$$||\vec{N}||^2 = \left((\partial_\theta r)^2 \sin^2\phi + r^2\cos^2\phi\right)\left((\partial_\phi r)^2\cos^2\phi + r^2\sin^2\phi - 2\partial_\phi r\cos\phi r\sin\phi\right)$$
$$+\cos^2\phi((\partial_\phi r)^2\sin^2\phi + r^2\cos^2\phi + 2\partial_\phi r\sin\phi r\cos\phi)\left((\partial_\theta r)^2 + r^2\right)$$
$$-2(\partial_\theta r)^2\sin\phi\cos\phi(\sin\phi\cos\phi((\partial_\phi r)^2 - r^2) + r\partial_\phi r(\cos^2\phi - \sin^2\phi))$$

$$= (\partial_\phi r)^2(\partial_\theta r)^2\sin^2\phi\cos^2\phi + r^2\sin^4\phi(\partial_\theta r)^2 - 2(\partial_\theta r)^2\partial_\phi r\cos\phi r\sin^3\phi$$
$$+(\partial_\phi r)^2\cos^4\phi r^2 + r^4\sin^2\phi\cos^2\phi - 2\partial_\phi r\cos^3\phi\sin\phi r^3$$
$$+(\partial_\phi r)^2(\partial_\theta r)^2\sin^2\phi\cos^2\phi + r^2\cos^4\phi(\partial_\theta r)^2 + 2\partial_\phi r\sin\phi r\cos^3\phi(\partial_\theta r)^2$$
$$+(\partial_\phi r)^2\sin^2\phi r^2\cos^2\phi + r^4\cos^4\phi + 2\partial_\phi r\sin\phi r^3\cos^3\phi$$
$$-2(\partial_\theta r)^2(\partial_\phi r)^2\sin^2\phi\cos^2\phi + 2(\partial_\theta r)^2\sin^2\phi\cos^2\phi r^2$$
$$-2(\partial_\theta r)^2\sin\phi\cos^3\phi r\partial_\phi r + 2(\partial_\theta r)^2\sin^3\phi\cos\phi r\partial_\phi r$$

$$= (\partial_\phi r)^2\sin^2\phi r^2\cos^2\phi + 2(\partial_\theta r)^2\sin^2\phi\cos^2\phi r^2$$
$$+r^4\cos^4\phi + r^2\cos^4\phi(\partial_\theta r)^2$$
$$+(\partial_\phi r)^2\cos^4\phi r^2 + r^4\sin^2\phi\cos^2\phi + r^2\sin^4\phi(\partial_\theta r)^2$$

$$= (\partial_\theta r)^2(r^2\cos^4\phi + 2\sin^2\phi\cos^2\phi r^2 + r^2\sin^4\phi)$$
$$+(\partial_\phi r)^2(\cos^4\phi r^2 + \sin^2\phi r^2\cos^2\phi)$$
$$+r^4(\cos^4\phi + \sin^2\phi\cos^2\phi)$$

Note that:

$$r^2 = r^2(\cos^2\phi + \sin^2\phi)^2 = (r\cos^2\phi + r\sin^2\phi)^2 = r^2\cos^4\phi + 2r^2\sin^2\phi\cos^2\phi + r^2\sin^4\phi$$

And:

$$\cos^4\phi r^2 + \sin^2\phi r^2\cos^2\phi = \cos^2\phi(r^2\cos^2\phi + r^2\sin^2\phi) = r^2\cos^2\phi$$

Same with:

$$r^4(\cos^4\phi + \sin^2\phi\cos^2\phi) = r^4\cos^2\phi(\cos^2\phi + \sin^2\phi) = r^4\cos^2\phi$$

So that the final result:

$$||\vec{N}||^2 = (\partial_\theta r)^2 r^2 + (\partial_\phi r)^2 r^2\cos^2\phi + r^4\cos^2\phi$$

$$= r^2((\partial_\theta r)^2 + \left((\partial_\phi r)^2 + r^2\right)\cos^2\phi)$$

Which implies that the modulus of $\vec{N}$ is

$$||\vec{N}|| = r\sqrt{(\partial_\theta r)^2 + ((\partial_\phi r)^2 + r^2)\cos^2\phi} \tag{8}$$

As the final step, computing dS gives

$$\mathrm{d}S = r(\theta,\phi)\sqrt{\left(\frac{\partial r}{\partial\theta}\right)^2 + \left(r^2(\theta,\phi) + \left(\frac{\partial r}{\partial\phi}\right)^2\right)\cos^2(\phi)}\,\mathrm{d}\phi\mathrm{d}\theta \tag{9}$$

Which is what it was wanted to be proved. Hence the proof is complete. ■